Updating Inertial Navigation Systems with VOR/DME Information

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Updating an inertial navigation system (INS) with VOR/DME information (from one or two stations) by means of a maximum-likelihood filter is shown to result in substantial improvements in navigational accuracy over that obtained by the use of a single VOR/DME (current practice). When continuously updating, the use of a high-quality INS (0.01°/hr gyro drift) instead of a low-quality INS (1.0°/hr gyro drift) does not substantially improve position accuracy. In-flight alignment (or realignment) of an INS to an accuracy comparable to that of ground alignment can be accomplished by using two DME's. Several reduced-order suboptimal filters were found to perform nearly optimally.

Introduction

THE primary navigation aid for civil aircraft flying in the airspace of most of the developed countries of the world is the VOR/DME system. The VOR (Very high-frequency Omni-Range) and the DME (Distance Measuring Equipment) enable onboard determination of an aircraft's bearing relative to north at the fixed ground station and slant range from the station, respectively.

Current use of the VOR/DME system involves primarily radial navigation, i.e., aircraft fly directly to or from the ground stations. However, some beginnings have been made in using the VOR/DME system for area navigation, i.e., use of the system without being restricted to fly directly to or from the ground stations.

The number of commercial airliners equipped with inertial navigation systems (INS's) is steadily increasing. The systems now onboard aircraft utilize a gyro-stabilized platform on which the accelerometers are mounted. The platform is aligned before takeoff to the desired orientation. Due to alignment errors and in-flight random disturbances such as gyro drift, scale-factor errors, and accelerometer bias errors, errors in the desired orientation of the platform increase with time. This results in increasing position and velocity errors. Thus, improving navigational accuracy by in-flight realignment of an INS is an interesting possibility.

Position errors are generally greater for area than for radial navigation. This comes about because the position error resulting from a VOR angular error increases with distance from the station; and an aircraft is farther, on the average, from the VOR stations for area than for radial navigation. Hence, improved navigational accuracy is required to obtain an accuracy for area navigation comparable to that of present-day radial navigation. The availability of an onboard computer to do the triangulation computations required for area navigation suggests the possibility of obtaining improved navigational accuracy by using the computer to implement a filter to combine VOR/DME information with the information from a dead-reckoning system. Since air data (airspeed and heading) are already available onboard nearly all aircraft and an increasing number of aircraft are equipped with inertial navigations systems, air and inertial data are the foremost choices of dead-reckoning information. The use of air data has been previously considered[1]; the use of inertial data is of concern here.

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Problem Statement

The use of external position information to update an INS has been previously considered. However, attention has been focused on the use of long-range radio systems such as LORAN and OMEGA. Because the accuracy of the position information derived from the VOR/DME system depends upon the relative location of the aircraft and the VOR/DME station as well as the number of VOR and DME stations used, the possibility of using VOR/DME information requires specific consideration. Although the use of VOR/DME information to update an INS has been mentioned in the literature,[2–6] no comprehensive study of the possibility of combining VOR/DME information and inertial data has been found.

Our objective here is to study the possibility of improving the accuracy of air navigation by combining VOR/DME information with inertial data by means of a maximum-likelihood filter. This is equivalent to in-flight alignment of the INS.‡ Since more than one VOR/DME station is nearly always in sight at jet altitudes, the simultaneous use of two VOR/DME stations and inertial data is considered. DeGroot and Larsen[7] have considered the use of two VOR's and two DME's (without inertial data). The possibilities of using 0, 1, or 2 VOR's and 0, 1, or 2 DME's with and without inertial data are considered in this work.

In-flight realignment, e.g., before or after a transoceanic leg of a flight, or periodically (every one or two hours) during a domestic flight, is also considered. Using VOR/DME information to simply reset the INS position display (no filtering involved) is compared with using the VOR/DME information to realign the

‡ Although alignment usually refers to physically rotating the INS platform to a desired orientation, in this work alignment simply means the estimation of the platform attitude errors. Compensation for these errors is then made in estimating velocity and position.

system. The possible improvement of navigational accuracy over land areas by using an INS continuously updated with VOR/DME information is investigated. Both radial and area flights are considered; the use of a high-quality INS is compared with the use of a low-quality system. The performances of several suboptimal filters, resulting when various states are neglected, are studied.

Error Model for the Inertial Navigation System

Three sets of coordinate axes enter into the INS error analysis: 1) platform coordinate axes, the axes lying along the nominal accelerometer sensitive axes, 2) locally-level coordinate axes, the axes representing the ideal alignment of the platform axes at the actual location of the platform, and 3) computer coordinate axes, the axes corresponding to the locally-level coordinate axes at the location indicated by the INS computer. For a perfectly operating system, the platform, locally-level, and computer sets of axes coincide. However, in actual operation these coordinate axes are generally rotated relative to each other through small angles.§

It can be shown[8,9] that the INS error equations may be put in the following form:¶

$$\delta\dot{\mathbf{R}} = \delta\mathbf{V} - \boldsymbol{\rho} \times \delta\mathbf{R} \tag{1}$$

$$\delta\dot{\mathbf{V}} = \boldsymbol{\alpha} - \omega_s^2 \delta\mathbf{R} - \boldsymbol{\psi} \times \mathbf{a} - (2\boldsymbol{\Omega} + \boldsymbol{\rho}) \times \delta\mathbf{V} + 3(\omega_s^2/R)\delta R\mathbf{R} \tag{2}$$

$$\dot{\boldsymbol{\psi}} = \boldsymbol{\varepsilon} - \boldsymbol{\omega} \times \boldsymbol{\psi} \tag{3}$$

where the time derivatives are taken in locally-level coordinates, and $\mathbf{R}$ = radius vector from the center of the Earth to the vehicle, R = magnitude of $\mathbf{R}$, $\delta\mathbf{R}$ = error in computing $\mathbf{R}$, i.e., position error, δR = magnitude of $\delta\mathbf{R}$, $\mathbf{V}$ = velocity of the vehicle relative to the Earth, $\delta\mathbf{V}$ = error in computing $\mathbf{V}$, $\boldsymbol{\rho}$ = angular velocity of the locally-level coordinates relative to the Earth, $\boldsymbol{\Omega}$ = angular velocity of the Earth relative to inertial space, $\omega_s^2 = g/R$, the square of the Schuler angular frequency, $\mathbf{g}$ = gravity vector (including centripetal acceleration), g = magnitude of $\mathbf{g}$, $\boldsymbol{\alpha}$ = accelerometer bias error, $\boldsymbol{\psi}$ = vector angle relating the platform and computer coordinates, $\boldsymbol{\varepsilon}$ = platform drift rate, $\boldsymbol{\omega} = \boldsymbol{\rho} + \boldsymbol{\Omega}$, and $\mathbf{a}$ = accelerometer output.

For the simulations performed here, $\mathbf{a}$ is taken to be the accelerometer output along a nominal flight path, i.e.,

$$\mathbf{a} = \dot{\mathbf{V}} + (2\boldsymbol{\Omega} + \boldsymbol{\rho}) \times \mathbf{V} - \mathbf{g} \tag{4}$$

where the differentiation is taken in locally-level coordinates and nominal values of $\mathbf{V}$, $\boldsymbol{\Omega}$, $\boldsymbol{\rho}$, and $\mathbf{g}$ are used.

For this study, a locally-level, north-pointing mechanization is used, i.e., the platform is maintained as closely as possible to locally-level, with respect to the Earth, with accelerometer sensitive axes pointing east (x), north (y), and up (z). Since this mechanization breaks down near the poles of the Earth, it is assumed that the vehicle does not operate at high latitudes.

Since the vertical channel of the INS is unstable (Ref. 8, Sec. 4.6), it is assumed for this study that altitude information is available from another source such as a barometric altimeter. Thus, only the two horizontal channels of the INS are considered. A further assumption is that the Earth is spherical and that gravitational equipotential surfaces associated with $\mathbf{g}$ are spherical. Although the nonspherical character of the Earth and its gravitational field must be taken into account during actual operation, for the simulations performed in this work, this assumption is considered reasonable.

Under the preceding assumptions, it follows that

$$R_x = R_y = 0, \quad R_z = R, \quad \delta R_z \cong \delta R \cong 0 \tag{5}$$

$$g_x = g_y = 0, \quad g_z = -g, \quad g = g_0(R_0^2/R^2) \tag{6}$$

$$\Omega_x = 0, \quad \Omega_y = \Omega \cos\lambda, \quad \Omega_z = \Omega \sin\lambda \tag{7}$$

$$\rho_x = -V_y/R, \quad \rho_y = V_x/R, \quad \rho_z = V_x/R \tan\lambda \tag{8}$$

where g_0 is the magnitude of $\mathbf{g}$ at the surface of the Earth ($\cong$32.2 ft/sec^2), R_0 is the mean radius of the Earth ($\cong$3440 naut miles), Ω is the Earth rotation rate ($\cong$15.04°/hr) and λ is the latitude. The subscript x, y, or z denotes the component along the x, y, or z axis of the associated vector quantity.

The platform attitude error $\boldsymbol{\phi}$, i.e., the vector angle relating the platform and locally-level, north-pointing (L) axes, is given by

$$\boldsymbol{\phi} = \boldsymbol{\psi} + \delta\boldsymbol{\theta} \tag{9}$$

where $\delta\boldsymbol{\theta}$ is the vector angle relating the computer and the L axes. For the mechanization used, the components of $\boldsymbol{\phi}$ are given by

$$\phi_x = \psi_x - \delta R_y/R, \quad \phi_y = \psi_y + \delta R_x/R, \quad \phi_z = \psi_z + (\delta R_x/R)\tan\lambda \tag{10}$$

where ϕ_x and ϕ_y are the platform tilts about the east and north axes, respectively; and ϕ_z is the platform azimuth error.

The east and north accelerometer errors (α_x and α_y) will now be modeled. Assuming a well calibrated high-quality INS operating in cruise conditions, the main error in the accelerometers is due to null shifts (or bias). This error can be modeled as an exponentially-correlated process.[10,11] A reasonable value for the standard deviation of this process is $10^{-4}g$ while the mean value is assumed to be zero. The correlation time is very long, a reasonable value being 10 hr. Hence, shaping filters which generate α_x and α_y are given by (see, e.g., Ref. 12, Sec. 11.4)

$$\dot{\alpha}_i = -(1/T_{\alpha_i})\alpha_i + (1/T_{\alpha_i})n_{\alpha_i}, \quad i = x, y \tag{11}$$

where n_{α_x} and n_{α_y} are independent white noise processes with**

$$E[n_{\alpha_i}] = 0, \quad E[n_{\alpha_i}(t+\tau)n_{\alpha_i}(t)] = 2T_{\alpha_i}\sigma_{\alpha_i}^2\delta(\tau) \tag{12}$$

The quantities σ_{α_x} and T_{α_x}, and σ_{α_y} and T_{α_y} are the standard deviation and correlation time of α_x and α_y, respectively.

The platform drift rate $\boldsymbol{\varepsilon}$ is due mainly to uncompensated gyro drift. Hence, gyro drift is the only source of gyro error that will be modeled. For our purposes, the gyro drift can be modeled reasonably well as an exponentially-correlated process.[10,11] For high-quality gyros, a typical value of the correlation time is 5 hr. Hence, the models for east gyro drift (ε_x), north gyro drift (ε_y), and vertical gyro drift (ε_z) are given by

$$\dot{\varepsilon}_i = -(1/T_{\varepsilon_i})\varepsilon_i + (1/T_{\varepsilon_i})n_{\varepsilon_i}, \quad i = x, y, z \tag{13}$$

where n_{ε_x}, n_{ε_y}, n_{ε_z} are independent white noise processes with

$$E[n_{\varepsilon_i}] = 0, \quad E[n_{\varepsilon_i}(t+\tau)n_{\varepsilon_i}(t)] = 2T_{\varepsilon_i}\sigma_{\varepsilon_i}^2\delta(\tau) \tag{14}$$

The quantities σ_{ε_x} and T_{ε_x}, σ_{ε_y} and T_{ε_y}, and σ_{ε_z} and T_{ε_z} are the standard deviation and correlation time of ε_x, ε_y, and ε_z, respectively.

Error Model for the VOR/DME System

Error models for the VOR and DME systems have been previously derived.[1] Both the VOR and DME errors can be separated into two components, a component with a long correlation time which can be modeled as a random bias and a component with a short correlation time which can be modeled as white noise. The mean values of all of these error components are essentially zero.

Letting b_V and e_V denote the VOR error components with a long and a short correlation time, respectively, we have

$$\dot{b}_V = 0 \tag{15}$$

$$E[e_V] = 0, \quad E[e_V(t+\tau)e_V(t)] = 2\sigma_{e_V}^2 T_{e_V}\delta(\tau) \tag{16}$$

where σ_{e_V} and T_{e_V} are the standard deviation and correlation time of e_V, respectively. A reasonable value for σ_{e_V} and σ_{b_V}, the standard deviation of e_V and b_V, respectively, is 1.0°. The correlation distance of e_V appears to be about one naut mile, which corresponds to a correlation time of 7.2 sec for an aircraft traveling at a speed of 500 knots.

Similarly, letting b_D and e_D devote the DME error components with a long and a short correlation time, respectively, we have

§ Hence, the theory of infinitesimal rotations is assumed to be applicable.

¶ A dot denotes differentiation with respect to time; a bold face quantity is a vector.

** $E[\ ]$ is the expected value function and $\delta(\tau)$ the Dirac delta function.

$$\dot{b}_D = 0 \quad (17)$$

$$E[e_D] = 0, \quad E[e_D(t+\tau)e_D(t)] = 2\sigma_{e_D}{}^2 T_{e_D} \delta(\tau) \quad (18)$$

where σ_{e_D} and T_{e_D} are the standard deviation and correlation time of e_D, respectively. Realistic values of σ_{e_D} and T_{e_D} are 0.1 naut mile and 3.6 sec, respectively. A reasonable value for σ_{b_D}, the standard deviation of b_D, is 0.14 naut mile.

The INS, VOR, and DME error models are summarized in Table 1. The correlation times shown are for an aircraft flying at a speed of 500 knots. The mean values and rms (root-mean-square) values are reasonable values for a "typical" VOR/DME station, assuming the use of quality receivers, i.e., the type used on jet airliners. The values of the INS error parameters shown in Table 1 are for a high-quality system. For a low-quality INS, the rms value of the gyro drifts is taken to be 1.0°/hr.

Table 1 Summary of INS, VOR, and DME error models

System	Error	Mean value	rms value	Correlation time	Model
INS	$\varepsilon_x, \varepsilon_y, \varepsilon_z$	0	0.01°/hr	5 hr	Exponentially correlated process
	α_x, α_y	0	$10^{-4}g$	10 hr	Exponentially correlated process
VOR	b_V	0	1.0°	hours	Random bias
	e_V	0	1.0°	7.2 sec	White noise
DME	b_D	0	0.14 naut mile	hours	Random bias
	e_D	0	0.1 naut mile	3.6 sec	White noise

System Model

A model for a system using an inertial navigation system and the information from two VOR/DME stations is given below:

$$\left.\begin{aligned}
\delta\dot{R}_x &= \rho_z \delta R_y + \delta V_x \\
\delta\dot{R}_y &= -\rho_z \delta R_x + \delta V_y \\
\delta\dot{V}_x &= -\omega_s{}^2 \delta R_x + (2\Omega_z + \rho_z)\delta V_y - a_z \psi_y + a_y \psi_z + \alpha_x \\
\delta\dot{V}_y &= -\omega_s{}^2 \delta R_y - (2\Omega_z + \rho_z)\delta V_x + a_z \psi_x - a_x \psi_z + \alpha_y \\
\dot{\psi}_x &= \omega_z \psi_y - \omega_y \psi_z + \varepsilon_x \\
\dot{\psi}_y &= -\omega_z \psi_x + \omega_x \psi_z + \varepsilon_y \\
\dot{\psi}_z &= \omega_y \psi_x - \omega_x \psi_y + \varepsilon_z \\
\dot{\varepsilon}_x &= -(1/T_{\varepsilon_x})(\varepsilon_x - n_{\varepsilon_x}) \\
\dot{\varepsilon}_y &= -(1/T_{\varepsilon_y})(\varepsilon_y - n_{\varepsilon_y}) \\
\dot{\varepsilon}_z &= -(1/T_{\varepsilon_z})(\varepsilon_z - n_{\varepsilon_z}) \\
\dot{\alpha}_x &= -(1/T_{\alpha_x})(\alpha_x - n_{\alpha_x}) \\
\dot{\alpha}_y &= -(1/T_{\alpha_y})(\alpha_y - n_{\alpha_y}) \\
\dot{b}_{V1} &= 0 \\
\dot{b}_{D1} &= 0 \\
\dot{b}_{V2} &= 0 \\
\dot{b}_{D2} &= 0
\end{aligned}\right\} \quad (19)$$

where the subscripts 1 and 2 indicate the VOR/DME station with which a quantity is associated. Equations (19) are in the state variable form

$$\dot{\mathbf{x}} = F\mathbf{x} + \mathbf{n} \quad (19a)$$

where F is a 16×16 matrix with many zeroes, and the vector $\mathbf{n}$ has many zeroes.

The VOR/DME station configuration is shown in Fig. 1. The vector $\mathbf{R}_{12}$ is defined by

$$\mathbf{R}_{12} \triangleq \mathbf{R}_1 - \mathbf{R}_2 \quad (20)$$

The easterly, northerly, and vertical components of $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_{12}$ are denoted by (x_1, y_1, h_1), (x_2, y_2, h_2), and (x_{12}, y_{12}, h_{12}), respectively. Denoting the VOR and DME measurements from station 1 and station 2 by V_1 and D_1, and V_2 and D_2, respectively, we have

$$\begin{bmatrix} V_1 \\ D_1 \\ V_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} \arctan(x_1/y_1) + b_{V1} \\ [x_1{}^2 + y_1{}^2 + h_1{}^2]^{1/2} + b_{D1} \\ \arctan[(x_1 - x_{12})/(y_1 - y_{12})] + b_{V2} \\ [(x_1 - x_{12})^2 + (y_1 - y_{12})^2 + (h_1 - h_{12})^2]^{1/2} + b_{D2} \end{bmatrix} + \begin{bmatrix} e_{V1} \\ e_{D1} \\ e_{V2} \\ e_{D2} \end{bmatrix} \quad (21)$$

The state varibles of the system are errors in the inertial and VOR/DME systems. If Eq. (21) is expanded in a Taylor series about a nominal flight path (the nominal values of x_1 and y_1 being denoted by $\bar{x}_1$ and $\bar{y}_1$, respectively) and only first-order terms are retained, the following measurement equations result:

$$\mathbf{z} = H\mathbf{x} + \mathbf{v}$$

i.e. (22)

$$\begin{bmatrix} \delta V_1 \\ \delta D_1 \\ \delta V_2 \\ \delta D_2 \end{bmatrix} = \left[\begin{array}{cc:c:c} \bar{y}_1/A, & -\bar{x}_1/A & & \\ \bar{x}_1/[A + h_1{}^2]^{1/2}, & \bar{y}_1/[A + h_1{}^2]^{1/2} & 0 & I \\ (\bar{y}_1 - y_{12})/B, & (x_{12} - \bar{x}_1)/B & (4 \times 10) & (4 \times 4) \\ \dfrac{\bar{x}_1 - x_{12}}{[B + (h_1 - h_{12})^2]^{1/2}}, & \dfrac{\bar{y}_1 - y_{12}}{[B + (h_1 - h_{12})^2]^{1/2}} & & \end{array}\right] \begin{bmatrix} \delta R_x \\ \delta R_y \\ \delta V_x \\ \delta V_y \\ \psi_x \\ \psi_y \\ \psi_z \\ \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \alpha_x \\ \alpha_y \\ b_{V1} \\ b_{D1} \\ b_{V2} \\ b_{D2} \end{bmatrix} + \begin{bmatrix} e_{V1} \\ e_{D1} \\ e_{V2} \\ e_{D2} \end{bmatrix}$$

with

$$A = \bar{x}_1{}^2 + \bar{y}_1{}^2, \quad B = (\bar{x}_1 - x_{12})^2 + (\bar{y}_1 - y_{12})^2 \quad (23)$$

where δV_1 and δD_1, and δV_2 and δD_2 denote the differences between the actual and nominal VOR and DME measurements from stations 1 and 2, respectively.

In the derivation of the measurement equations, it was assumed that the L axes, i.e., the set of axes with origin at the location of the aircraft and axes pointing east, north, and up, and the

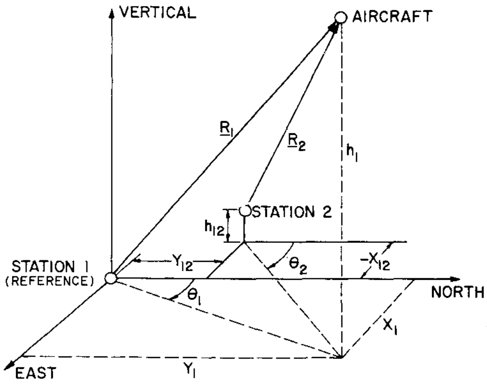

Fig. 1 VOR/DME station configuration.

reference station coordinate axes (see Fig. 1) are parallel when in fact they are not, due to the curvature of the Earth's surface. However, since an aircraft can use a VOR/DME station which is at most 200 naut miles away, the true and coordinate axes are rotated relative to each other through small angles when not at high latitudes. Hence, the differences between the components of $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_{12}$ in true and station coordinates are small and are therefore neglected. In other words, in the simulations, flight paths consisting of a series of straight line segments rather than true great circle paths are flown.

Letting t_0 denote the time that the system is initialized, it follows from the models derived previously that

$$E[\mathbf{n}(t)] = E[\mathbf{v}(t)] = E[\mathbf{n}(t)\mathbf{v}^T(\tau)] = 0 \tag{24}$$

$$E[\mathbf{x}(t_0)] = E[\mathbf{x}(t_0)\mathbf{n}^T(t)] = E[\mathbf{x}(t_0)\mathbf{v}^T(t)] = 0 \tag{25}$$

$$\begin{aligned} E[\mathbf{n}(t+\tau)\mathbf{n}^T(t)] &= Q\delta(\tau) \\ &= \text{diag}\left\{0,0,0,0,0,0,0,\frac{2\sigma_{\varepsilon_x}^2}{T_{\varepsilon_x}},\frac{2\sigma_{\varepsilon_y}^2}{T_{\varepsilon_y}},\frac{2\sigma_{\varepsilon_z}^2}{T_{\varepsilon_z}},\right. \\ &\qquad \left.\frac{2\sigma_{\alpha_x}^2}{T_{\alpha_x}},\frac{2\sigma_{\alpha_y}^2}{T_{\alpha_y}},0,0,0,0\right\}\delta(\tau) \end{aligned} \tag{26}$$

$$\begin{aligned} E[\mathbf{v}(t+\tau)\mathbf{v}^T(t)] &= R\delta(\tau) \\ &= \text{diag}\{2T_{ev1}\sigma_{ev1}^2, 2T_{eD1}\sigma_{eD1}^2, \\ &\qquad 2T_{ev2}\sigma_{ev2}^2, 2T_{eD2}\sigma_{eD2}^2\}\delta(\tau) \end{aligned} \tag{27}$$

The error covariance matrix P, at time t_0 is given by

$$P(t_0) = E[\mathbf{x}(t_0)\mathbf{x}^T(t_0)] \tag{28}$$

where all the elements are zero except those on the main diagonal, and possibly $P_{12}(t_0)$ (the element in the first row and second column) and $P_{21}(t_0)$. In particular

$$\begin{aligned} \text{diag}\,P(t_0) = \{&\sigma_{R_x}^2(t_0), \sigma_{R_y}^2(t_0), \sigma_{V_x}^2(t_0), \sigma_{V_y}^2(t_0), \\ &\sigma_{\psi_x}^2(t_0), \sigma_{\psi_y}^2(t_0), \sigma_{\psi_z}^2(t_0), \sigma_{\varepsilon_x}^2, \sigma_{\varepsilon_y}^2, \sigma_{\varepsilon_z}^2, \\ &\sigma_{\alpha_x}^2, \sigma_{\alpha_y}^2, \sigma_{bV1}^2, \sigma_{bD1}^2, \sigma_{bV2}^2, \sigma_{bD2}^2\} \end{aligned} \tag{29}$$

$$\sigma_{R_xR_y}^2 = P_{12}(t_0) = P_{21}(t_0) = E[\delta R_x(t_0)\delta R_y(t_0)] \tag{30}$$

In Eqs. (29) and (30), the initial position error statistics may be those associated with a VOR/DME reading[1] or those determined from another estimate of position. The variances of the velocity and platform attitude errors will depend on the state of the INS when the filter is initialized, and the values of the remainder of the variances appearing in Eq. (29) are readily determined from the error models derived previously.

Discretization of the System Model

For the purposes of simulation, the continuous, time-varying, linear system under consideration will be approximated by a multistage system. Thus, a multistage process described by

$$\mathbf{x}_{i+1} = \Phi_i\mathbf{x}_i + \Gamma_i\mathbf{n}_i, \quad i = 0,1,2,\ldots \tag{31}$$

is sought such that

$$\left.\begin{aligned} &\mathbf{x}_0 \cong \mathbf{x}(t_0) \\ &\mathbf{x}_i \cong \mathbf{x}(t_i), \quad t_i = t_{i-1} + \Delta t, \quad i = 1,2,\ldots \end{aligned}\right\} \tag{32}$$

where ΔT is a time increment. The continuous process described by Eq. (19) can be approximated to first order in ΔT by the process described by Eq. (31) if (Ref. 12, Sec. 11.5)

$$\Phi_i = [I + F(t_i)\Delta T] \tag{33}$$

$$\Gamma_i = \Delta T \tag{34}$$

$$\mathbf{n}_i = \mathbf{n}(t_i), \quad t_i \leqq t < t_{i+1} \tag{35}$$

with

$$E[\mathbf{n}_i\mathbf{n}_j^T] = Q'\delta_{ij} = Q/\Delta T\,\delta_{ij} \tag{36}$$

where δ_{ij}, the Kronecker delta function, is equal to zero unless $i = j$, in which case it equals one. Furthermore, the discrete approximation to the continuous measurement system described by Eqs. (22) is given by

$$\mathbf{z}_i = H_i\mathbf{x}_i + \mathbf{v}_i, \quad i = 0,1,2,\ldots, \tag{37}$$

with

$$\mathbf{v}_i = \mathbf{v}(t_i), \quad t_i \leqq t < t_{i+1} \tag{38}$$

where

$$E[\mathbf{v}_i\mathbf{v}_j^T] = R'\delta_{ij} = R/\Delta T\,\delta_{ij} \tag{39}$$

The approximation holds only if ΔT is small compared to the characteristic times of the system.

Filter for Combining VOR/DME and Inertial Data

The Kalman filter equations[13] for the linear multistage approximation described by Eqs. (31–39) are:

Time update:

$$\hat{\mathbf{x}}_{i+1} = \Phi_i\hat{\mathbf{x}}_{i+} \tag{40}$$

$$P_{i+1} = \Phi_i P_{i+}\Phi_i^T + \Gamma_i Q'\Gamma_i^T \tag{41}$$

Measurement update:

$$\hat{\mathbf{x}}_{i+} = \hat{\mathbf{x}}_i + K_i(\mathbf{z}_i - H_i\hat{\mathbf{x}}_i) \tag{42}$$

$$P_{i+} = (I - K_iH_i)P_i(I - K_iH_i)^T + K_iR'K_i^T \tag{43}$$

$$K_i = P_iH_i^T(H_iP_iH_i^T + R')^{-1} \tag{44}$$

where $\hat{\mathbf{x}}_i$ and P_i, and $\hat{\mathbf{x}}_{i+}$ and P_{i+} denote the state estimate and covariance matrix at stage i before and after processing the measurement at stage i, respectively, and I is the 16×16 identity matrix.

Note that in the absence of measurements, the propagation of the state estimate and covariance matrix are described by Eqs. (40) and (41). This would correspond to the propagation of errors in unaided-inertial operation.

Combining VOR/DME information from two stations without inertial data reduces to the problem of finding the maximum-likelihood estimates of x_1 and y_1 (see Fig. 1). An estimator that performs this task is designed in Ref. 1.

The multistage filter described by Eqs. (40–44) estimates the easterly and northerly components of the position and velocity errors, the angles of rotation about the east, north, and vertical axes which relate the platform and computer axes, the gyro drift in each of the three gyros, the bias errors in the east and north accelerometers, and the bias in each VOR and each DME. In deriving the measurement equations, the rotation between the true and reference station coordinate axes, due to the curvature of the Earth, is neglected. Upon tuning in a new VOR/DME station, the associated VOR and DME biases must be reinitialized.

Simulation Studies

A computer program was written to calculate the error covariance matrix for any nominal flight path consisting of a series of straight line segments. The inputs to the program are: a) the model parameters, b) the latitude, longitude, and altitude

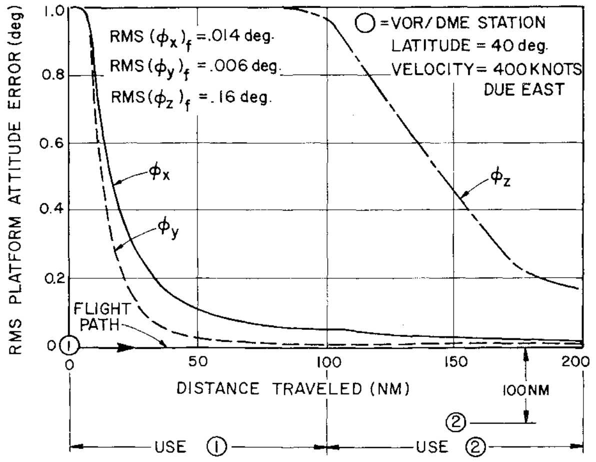

Fig. 2 Platform attitude errors for in-flight alignment using the information from one VOR/DME.

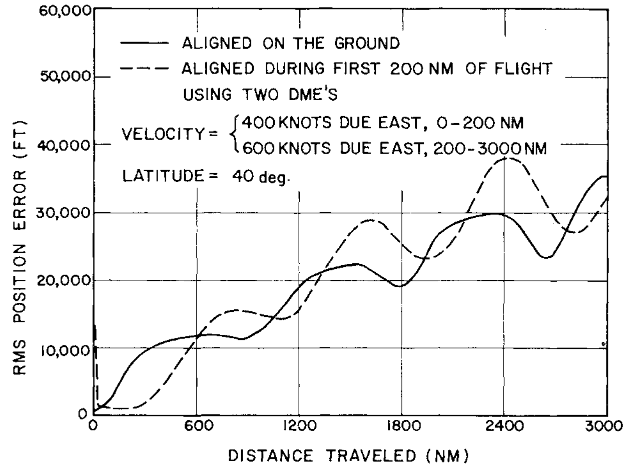

Fig. 4 The rms position errors resulting during an unaided-inertial flight preceded by ground and in-flight alignment.

of each of the VOR/DME stations to be used, and c) the latitude and longitude of each of the switching points, i.e., points along the flight path where the aircraft tunes in a new VOR/DME station or changes its speed or directional heading. The outputs of the program are the rms errors in the estimates of the states.

In-Flight Alignment

The possibility of performing the fine alignment of the INS platform while in the air by using VOR/DME information was investigated. Figure 2 shows the rms errors in the estimates of the platform attitude (ϕ_x = tilt about east axis, ϕ_y = tilt about north axis, and ϕ_z = aximuth error) for a 30-min flight (200 naut miles at 400 knots) using one VOR/DME to update a high-quality INS (0.01°/hr gyro drift). In Fig. 3 these same errors are shown for a flight using two DME's. The time between measurement updates was 60 sec and the altitude was 33,000 ft. The present method of ground alignment takes about 15 min with the final rms error in the platform tilts, rms $(\phi_x)_f$ and rms $(\phi_y)_f$, equal to about 0.005° and the final rms azimuth error, rms $(\phi_z)_f$, equal to about 0.05°. Thus, the accuracy of in-flight alignment using two DME's is about the same as for ground alignment, whereas in-flight alignment using one VOR/DME is less accurate by a factor of 2 or 3. In Fig. 4 the rms position errors during an unaided-inertial flight preceded by ground alignment are compared with errors during a flight with in-flight alignment (using two DME's, the station configuration being that of Fig. 3).

In-flight alignment using combinations of VOR/DME information other than the two previously discussed were also considered. The use of only the DME measurements during the flight of Fig. 2 yields much less accurate alignment than when both the VOR and DME measurements are used. The addition of VOR measurements during the flight of Fig. 3 does not significantly improve the accuracy of alignment.

Station configurations other than the two simple configurations of Figs. 2 and 3 were also studied. It was found that more complex configurations with more frequent switching between stations do not result in a significant improvement in alignment accuracy. Note that the results of Figs. 2 and 3 are not restricted to the specific flight paths considered. For example, if during a 30-min flight, the information from one VOR/DME is used where the line-of-sight to the station used during half of the flight is, in general, orthogonal to the line-of-sight to the station used during the other half of the flight, the resulting alignment will essentially be that of Fig. 2. Similarly, if the information from two DME's is used where the crossing angle between the lines-of-sight from the aircraft to the stations lies between 60° and 120°, the resulting alignment will be about the same as that of Fig. 3.

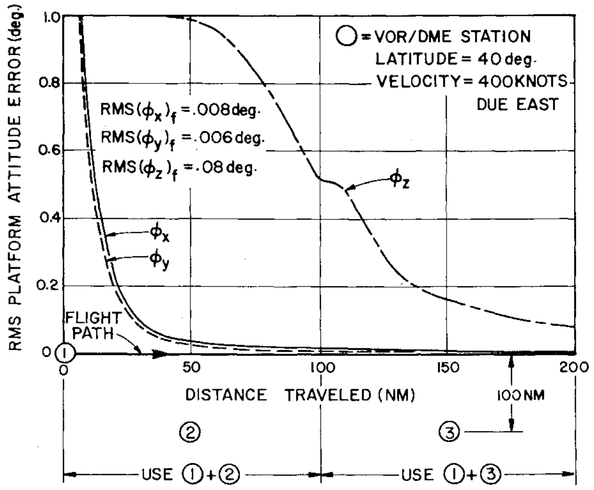

Fig. 3 Platform attitude errors for in-flight alignment using the information from two DME's.

Periodic Updating

The use of VOR/DME information to update a high-quality INS during a 5-hr transcontinental flight was investigated. The system is assumed to be initially aligned on the ground with a 0.005° rms error in the platform tilts and a 0.05° rms azimuth error. Furthermore, the initial position and velocity errors are taken to be 0.1 naut mile and 0.1 knot, respectively, in both the easterly and northerly directions. The altitude is 33,000 ft and the time between measurement updates is 2 min. The rms position errors are shown in Fig. 5 for the unaided-inertial case as well as for the cases where the system is realigned twice during the flight (from 800 to 1000 naut miles and from 1800 to 2000 naut miles). Realignment is done using one VOR/DME with the station configuration of Fig. 2 and using two DME's with the station configuration of Fig. 3. Obviously, periodic realignment of the INS results in more accurate position information than that resulting from unaided-inertial operation. There is not much difference between the errors resulting from the use of one VOR/DME and the use of two DME's.

Also shown in Fig. 5 are the position errors resulting if two position resets are performed during the flight (at 950 and 1950 naut miles). Here position reset means that the position display is reset to correspond to a position fix from two DME's (whose lines-of-sight are perpendicular). Thereafter, the increments of

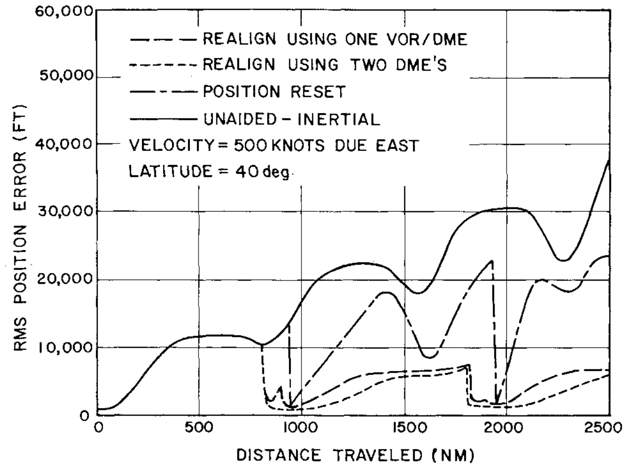

Fig. 5 The rms position errors for realignment, position reset, and unaided-inertial modes of operation during a transcontinental flight.

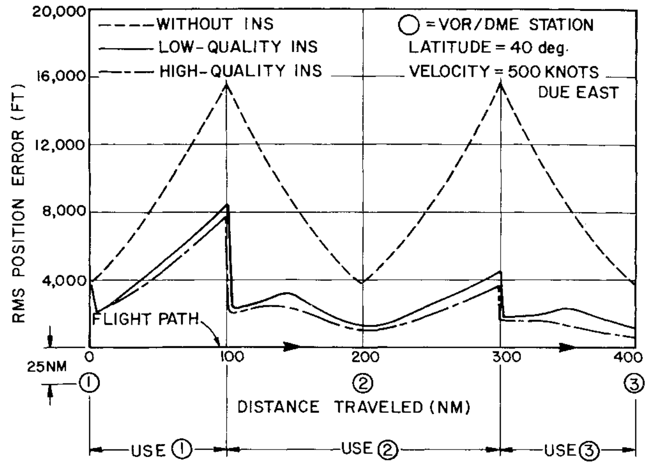

Fig. 7 The rms position errors for an area flight using one VOR/DME with a high-quality INS, a low-quality INS, and without an INS.

position change calculated by the INS are simply added to the display. The position used in the routine which integrates the outputs of the accelerometers is not changed because such a change causes large amplitude oscillations in the rms position and velocity error histories. Position resets must be performed frequently (every few minutes) to obtain position accuracy comparable to the case where the system is realigned. Obviously, position resets do not result in decreased velocity or platform drift errors.

Continuous Updating

Since the VOR/DME system provides nationwide coverage in the U.S., there exists the possibility of continuously updating an INS with VOR/DME information during domestic flights. For such flights it may be possible to use a low-quality INS instead of the high-quality system required for unaided-inertial operation.

In Fig. 6, the rms position errors for a radial flight using the information from one VOR/DME with a high-quality INS (0.01°/hr gyro drift), a low-quality INS (1.0°/hr gyro drift), and without an INS are shown. These same error histories are shown in Fig. 7 for an area flight. The time between measurement updates is 90 sec and the altitude is 33,000 ft. The rms value of the initial position error is that of a VOR/DME reading taken at the beginning of the flight while the rms velocity error is 10 knots in the northerly and easterly directions. The rms values of the initial platform tilts and azimuth error are 0.05 and 0.1°, and 0.5 and 5.0° for the high-quality and low-quality systems, respectively. As can be seen in Figs. 6 and 7, there is considerable improvement in position accuracy when inertial data are added to the information from one VOR/DME. However, the improvement in position accuracy when using a high-quality INS is not much greater than when using a low-quality INS.

The sharp decreases in rms position errors in Figs. 6 and 7 at the points where the aircraft switches from one VOR/DME station to another are of interest. The decreases which occur in the radial flight (as well as a portion of the decreases for the area flight) are due to a transient effect which is introduced when, upon tuning in a new station, the variances of the biases are reinitialized and the offdiagonal terms involving the biases are set to zero in the covariance matrix. The larger decreases occurring in the area flight are explained by the fact that the DME position information is more accurate than the VOR position information (except when very near the station), and the fact that the lines-of-sight to the new and old stations at the switching points for the area flight (Fig. 7) are not parallel.

In Fig. 8, the rms position errors are shown for various combinations of the information from two VOR/DME's and data from a low-quality INS. The use of a high-quality INS yields only a small improvement over the position accuracy

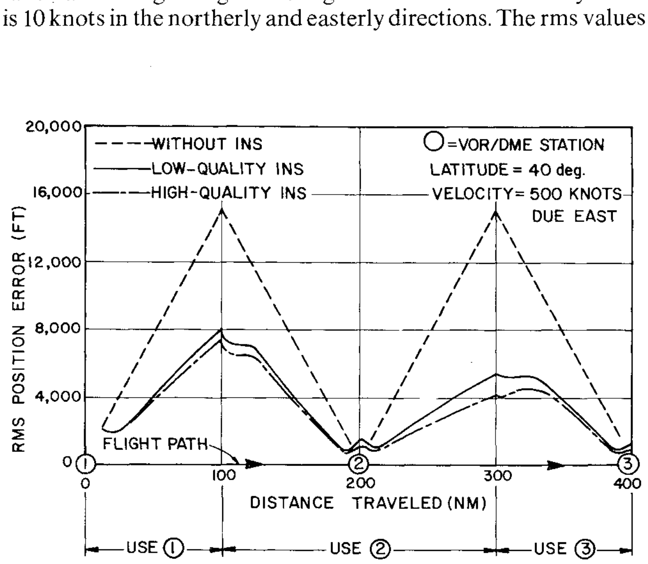

Fig. 6 The rms position errors for a radial flight using one VOR/DME with a high-quality INS, a low-quality INS, and without an INS.

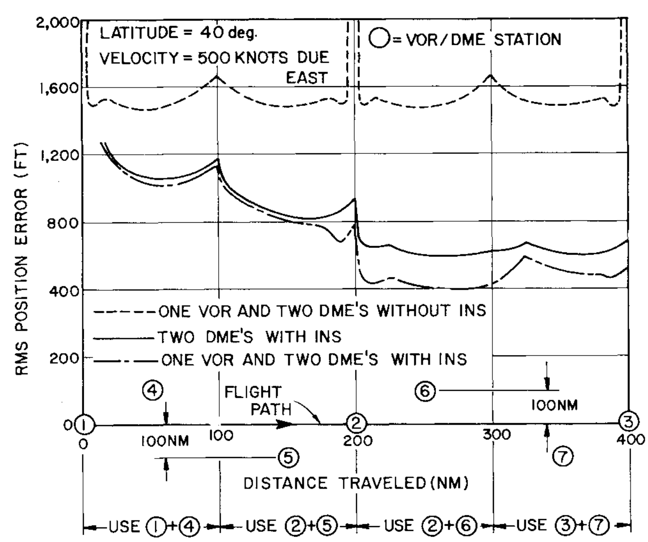

Fig. 8 The rms position errors for various combinations of the information from two VOR/DME's and data from a low-quality INS.

Table 2 Approximate factors of improvement in rms position accuracy over the use of a single VOR/DME for various combinations of VOR/DME information and data from a low-quality INS

Combination of information	Factor of improvement
1 VOR, 1 DME, INS (radial flight)	2.8
1 VOR, 1 DME, INS (area flight)	3.5
1 (or 2) VOR's, 2 DME's	9
2 DME's, INS	24
1 (or 2) VOR's, 2 DME's, INS	37

obtained when using a low-quality INS. When a second VOR is added to the case of one VOR, two DME's and inertial data, the decrease in the rms position error is negligible.

Factors of improvement in rms position error over the use of a single VOR/DME are shown in Table 2 for various combinations of information from two VOR/DME's and data from a low-quality INS. These factors were calculated for a point halfway between the second and third stations because at this point the error for the case of using a single VOR/DME is maximum; and hence, the factor of improvement at this point is of prime importance. The case where two DME's are used without inertial data is not included in Table 2 because there are generally two position fixes possible in this case. However, the addition of a VOR measurement to two DME measurements resolves the ambiguity, although it does not substantially improve the accuracy of the position fixes. From these results, it is seen that the navigational accuracy resulting from the use of a given combination of VOR/DME information is improved by roughly a factor of 3 or 4 by the addition of inertial data.

When using the information from one VOR/DME to update a low-quality INS, rms velocity errors of 7 or 8 knots can be expected, whereas, when using a high-quality INS, velocity errors of 3 or 4 knots occur. The use of the information from two VOR/DME stations to update an INS yields rms velocity errors of roughly one and two knots for a high-quality and a low-quality system, respectively.

If the flight paths of Figs. 6–8 were extended and similar configurations of stations were encountered, the error histories between the second and third stations would repeat. Insight into the use of air vs inertial data can be obtained by comparing these results with those of Ref. 1.

Suboptimal Filters

For the alignment or realignment of a high-quality INS (which takes from 20 to 30 min), the suboptimal filter resulting when the gyro drifts, accelerometer errors, and DME biases are neglected performs nearly optimally.[9] Hence, when using one VOR/DME, the alignment accuracy shown in Fig. 2 can be obtained by using an eighth-order filter, whereas, when using two DME's, the alignment accuracy shown in Fig. 3 is obtainable with a seventh-order filter. This seventh or eighth order suboptimal filter also behaves nearly optimally when using VOR/DME information to periodically update a high-quality INS.

When using VOR/DME information to continuously update an INS, the performances of the suboptimal filters resulting when certain VOR and DME biases are neglected were investigated. For the case where information from one VOR/DME and inertial data from a low-quality INS are used (for the flight path shown in Fig. 6), the performance degradation†† of the suboptimal filter resulting when the DME bias is neglected is less than 1%, whereas if both the VOR and DME biases are neglected, the performance degradation is about 125%. For the case of using two DME's with inertial data (for the flight path of Fig. 8), the performance degradation resulting when both DME biases are neglected is approximately 100%. In general, neglecting VOR biases results in great performance degradation, while neglecting DME biases does not. Note that although the 100% degradation stated for the case of two DME's and inertial data is a significant percentage, the degradation in terms of feet of rms position error is small (see Fig. 8). The above performance degradations were calculated at the midpoint between the second and third stations.

†† In this context, performance degradation means the increase in rms position error over that resulting from the optimal filter.

Conclusions

Combining VOR/DME information (from one or two stations) with inertial data by means of a maximum-likelihood filter results in substantial improvements in navigational accuracy over that obtained by the use of a single VOR/DME, as is the current practice.

The use of a low-quality INS with one VOR/DME reduces the rms position error by a factor of roughly 3, and yields rms velocity errors of about 8 knots. When using VOR/DME information continuously throughout the flight, the use of a high-quality INS (0.01°/hr gyro drift) instead of a low-quality INS (1.0°/hr gyro drift) does not substantially improve position accuracy, but does reduce the rms velocity error to about 4 knots. Some of the improvement due to the use of inertial data results from the ability of the filter to estimate the bias errors associated with the VOR/DME measurements. Estimates of VOR bias error are better for area than for radial flights while the opposite is true for the DME bias error.

The use of information from two VOR/DME stations with inertial data yields large factors of improvement (roughly 25 to 35) in rms position accuracy over the use of a single VOR/DME station. In general, the rms position error obtained when using a given combination of the VOR/DME information from two stations is decreased by a factor of about 3 or 4 by the addition of inertial data. When using information from two VOR/DME stations with data from a high-quality and a low-quality INS, the velocity errors were about 1 and 2 knots, respectively. As far as position and velocity accuracy are concerned, at most one VOR station need be used.

Accurate in-flight alignment of an INS platform by using VOR/DME information is possible. The accuracy of in-flight alignment using two DME's is about the same as for ground alignment, whereas, when using one VOR/DME, alignment is less accurate by a factor of 2 or 3. In-flight alignment takes about 30 min. Although initial in-flight alignment of inertial systems onboard commercial aircraft is probably not needed, the in-flight realignment of the system before a transoceanic flight (e.g., from Los Angeles to London or Chicago to Hawaii) results in significant improvement in navigational accuracy. This might permit a reduction of separation requirements over the North Atlantic routes. Also, realignment of the system after a transoceanic flight could prove useful since the result would be accurate position, velocity, and attitude information in the terminal area and for approach and landing.

Periodic realignment (every one or two hours) of a high-quality INS during a transcontinental flight results in significant improvements in position and velocity errors over unaided-inertial operation or the use of periodic position display resets. The use of a periodically-realigned, high-quality INS would make it possible to fly precise great circle routes between cities since the locations of VOR/DME stations would not substantially restrict the flight paths (as they do when using a continuously-updated INS). If the final realignment is performed just prior to entering the terminal area, accurate position, velocity, and attitude information would be available for approach and landing without reliance upon VOR/DME information.

In general, when combining VOR/DME information with inertial data, the suboptimal filter resulting when the DME biases are neglected performs nearly optimally. However, neglecting VOR biases results in great performance degradation. During the period of time that it takes for in-flight alignment (or realignment) of a high-quality INS (20 to 30 min), the suboptimal filter resulting when the gyro drifts, accelerometer errors, and DME biases are neglected performs nearly optimally. Hence, alignment can be accomplished by using an eighth-order filter when using one VOR/DME, or a seventh-order filter when using two DME's.

The quantitative results presented here are, of course, dependent upon the validity of the error models assumed. In particular, the VOR/DME error model proposed here has not been as thoroughly checked experimentally as the inertial navigation system model.

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Study of Vortex Rings Using a Laser Doppler Velocimeter

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Measurements of the axial and radial velocity distributions in vortex rings were made using a two-component laser Doppler velocimeter. The rings were generated by pulsing air through a sharp-edged orifice using a loudspeaker. The strength and vortex core size could be controlled somewhat by the duration and amplitude of the pulse. From detailed surveys of the velocity field, both the circulation and vorticity distribution were found for two different rings: one with a relatively thick core, the other with a thin core. The vorticity was found to be rather concentrated for both rings. Streamlines were also calculated and compared with observations. Vortex rings were found to be unstable to azimuthal perturbations; the observed mode number and growth rate are in reasonable agreement with theory.

Nomenclature

a = vortex core radius
n = number of unstable waves
r, z = coordinates
ρ, θ = polar coordinates at $r = r_0$
r_0 = vortex ring radius
U, V = velocities in r and z directions, respectively
U_0 = velocity of vortex ring
$\tilde{V}$ = $\Gamma_0/4\pi r_0$
v_0 = swirl velocity in ρ, θ coordinates
Γ = circulation
Γ_0 = total circulation of vortex ring
ψ = streamfunction
ω = azimuthal component of vorticity
ω_0 = zero-order vorticity
ω_1 = first-order vorticity

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Index category: Subsonic and Transonic Flow.

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Introduction

ALTHOUGH the fundamental interest in vortex rings has led to many flow visualization studies, there have been relatively few attempts to measure the detailed structure of the ring. This is due in part to the sensitivity of flows with concentrated vorticity to the insertion of a probe.

Tracer techniques were attempted by Krutzsch[1] in 1939 using aluminum flakes and Maxworthy[2] in 1970 using hydrogen bubbles, but due to strong axial gradients and associated radial velocities their results were inaccurate. Hot wire measurements[3] are hampered by probe interference and the difficulty of calibrat-